JOURNAL OF COMPUTATIONAL PHYSICS 77, 166–182 (1988)

Line Relaxation for Spectral Multigrid Methods

WILHELM HEINRICHS

Mathematisches Institut der Universität Düsseldorf,
4000 Düsseldorf 1, Federal Republic of Germany

Received March 18, 1987; revised June 12, 1987

Spectral multigrid methods for elliptic problems with Dirichlet and periodic boundary conditions are examined. The multigrid convergence is improved by certain line relaxation techniques which are suggested by a smoothing analysis. Furthermore several choices of relaxation parameters are compared. Numerical results are presented showing the gain in efficiency and accuracy over previous methods. © 1988 Academic Press, Inc.

1. INTRODUCTION

Spectral methods [5, 11] give very accurate approximations for smooth solutions of elliptic problems with relatively few degrees of freedom. The matrices involved are full (and nonsymmetric) and efficient iterative methods for the solution of the spectral systems are necessary. Above all pseudospectral (or collocation) methods can be implemented very efficiently with fast Fourier transforms [15, 16].

Zang *et al.* [19, 20] introduced multigrid techniques for the fast solution of spectral problems. Brandt *et al.* [2] have significantly improved spectral multigrid methods for periodic elliptic problems. Streett *et al.* [13] investigated combined Dirichlet and periodic problems and applied them to the transonic flow. They used alternate direction implicit (ADI) methods for relaxation. We achieved some improvements for both Dirichlet and combined Dirichlet/periodic problems. This was done by employing certain (alternating) line relaxation techniques (see also [2, 14, 17]). Furthermore we show that minimal residual relaxation [3] speeds up the convergence and also yields good results for problems far removed from Poisson's equation.

In Sections 2 and 3 we give the pseudospectral and finite difference discretizations for the Dirichlet problem. The smoothing analysis of Section 4.1 reveals the anisotropic behaviour of the discrete problems and suggests suitable defect corrections. For this purpose different incomplete LU-decompositions [9, 17] (applied in an alternating manner) and alternating zebra line relaxation (ZLR) [14] are described in Section 4.2. Furthermore several choices of relaxation parameters are discussed in Section 4.3. The numerical results in Section 5 show the improved convergence of our methods. In Section 6 some examples introduced by Haidvogel and Zang [6] are solved using full multigrid techniques [14]. As shown in Section 7 the

0021-9991/88 $3.00

preceding ideas can also be successfully adopted to combined Dirichlet and periodic problems. Here we investigate both rectangular and annular Chebyshev–Fourier methods.

Altogether the results presented show the effectiveness of a suitable preconditioning. They further substantiate the usefulness of spectral multigrid methods.

2. Pseudospectral Discretization

We consider the elliptic problem

$$Lu = -(au_x)_x - (bu_y)_y = f \tag{2.1}$$

on the region $\Omega = (-1, 1)^2$ with the Dirichlet boundary condition $u = g$ on $\partial\Omega$. Hereby a, b, f denote given functions.

For the discretization products of Chebyshev polynomials $t_p(x)\, t_q(y) = \cos(p \arccos x) \cos(q \arccos y)$ for $p, q = 0, 1, ..., N$ are employed. The grid points are

$$(x_i, y_j) = \left(\cos\frac{i\pi}{N}, \cos\frac{j\pi}{N}\right) \qquad \text{for} \quad i, j = 0, 1, ..., N.$$

We introduce the grids

$$\bar{\Omega}_N = \{(x_i, y_j): i, j = 0, ..., N\}, \qquad \Omega_N = \Omega \cap \bar{\Omega}_N,\ \partial\Omega_N = \partial\Omega \cap \bar{\Omega}_N.$$

By $(i, j) \in I_{\Omega_N}$ (resp. I_{Ω_N} or $I_{\partial\Omega_N}$) we mean that $(x_i, y_j) \in \bar{\Omega}_N$ (resp. Ω_N or $\partial\Omega_N$). $G(\bar{\Omega}_N)$ (resp. $G(\Omega_N)$ or $G(\partial\Omega_N)$) denote the set of grid functions defined on $\bar{\Omega}_N$ (resp. Ω_N or $\partial\Omega_N$). For the components of the grid functions $v_N \in G(\bar{\Omega}_N)$ we use the abbreviation $v_N^{i,j} = v_N(x_i, y_j)$ $((i, j) \in I_{\bar{\Omega}_N})$. Let the fixed grid functions $f_N \in G(\Omega_N)$, $g_N \in G(\partial\Omega_N)$ be defined by $f_N^{i,j} = f(x_i, y_j)$, $((i, j) \in I_{\Omega_N})$, $g_N^{i,j} = g(x_i, y_j)$ $((i, j) \in I_{\partial\Omega_N})$. The pseudospectral discretization of (2.1) leads to a discrete problem of the form

$$L_{sp} u_N = f_N \text{ on } \Omega_N, \qquad u_N = g_N \text{ on } \partial\Omega_N,$$

where L_{sp} denotes the known spectral matrix as introduced in [19].

3. The Finite Difference Discretization

In connection with the defect correction we need a finite difference discretization of the operator L. For this purpose we introduce the five point difference star

$$L_{FD}^{i,j} = \begin{bmatrix} 0 & \gamma_{0,1}^{i,j} & 0 \\ \gamma_{-1,0}^{i,j} & \gamma_{0,0}^{i,j} & \gamma_{1,0}^{i,j} \\ 0 & \gamma_{0,-1}^{i,j} & 0 \end{bmatrix} \qquad ((i, j) \in I_{\Omega_N}) \tag{3.1}$$

with

$$\gamma_{k,l}^{i,j} = -\beta\beta_{k,l}^{i,j},\ \beta = 1/(2\sin(\pi/2N)\sin(\pi/N))$$

and

$$\begin{aligned}
\beta_{0,1}^{i,j} &= b(x_i, (y_j + y_{j+1})/2)/(\sin((j+1/2)\pi/N)\sin(j\pi/N)),\\
\beta_{0,-1}^{i,j} &= b(x_i, (y_j + y_{j-1})/2)/(\sin((j-1/2)\pi/N)\sin(j\pi/N)),\\
\beta_{-1,0}^{i,j} &= a((x_i + x_{i+1})/2, y_j)/(\sin((i+1/2)\pi/N)\sin(i\pi/N)),\\
\beta_{1,0}^{i,j} &= a((x_i + x_{i-1})/2, y_j)/(\sin((i-1/2)\pi/N)\sin(i\pi/N)),\\
\beta_{0,0}^{i,j} &= -(\beta_{-1,0}^{i,j} + \beta_{1,0}^{i,j} + \beta_{0,1}^{i,j} + \beta_{0,-1}^{i,j}).
\end{aligned}$$

Corresponding difference operators L_{FD}^N: $G(\bar{\Omega}_N) \to G(\Omega_N)$ and $\hat{L}_{FD}^N$: $G(\Omega_N) \to G(\Omega_N)$ are defined by

$$(L_{FD}^N v_N)^{i,j} = L_{FD}^{i,j} v_N,\ (L_{FD}^N w_N)^{i,j} = L_{FD}^{i,j} w_N^0 \qquad ((i,j) \in I_{\Omega_N})$$

with $v_N \in G(\bar{\Omega}_N)$, $w_N \in G(\Omega_N)$ and $w_N^0 = w_N(\Omega_N)$, $w_N^0 = 0$ $(\partial\Omega_N)$.

The above discretization is the usual five point star for the Chebyshev–Lobatto points where addition theorems are used to get numerically stable formulas. Although the collocation points are not equidistant it is still second order (see [8]).

4. Relaxation Scheme

Central to the multigrid method is the relaxation scheme used to smooth the error on each grid. We use a Richardson (or Euler) scheme [18] combined with defect correction.

If some approximation $\tilde{u}_N \in G(\bar{\Omega}_N)$, $\tilde{u}_N = g_N(\partial\Omega_N)$ of u_N is given, the calculation of a new approximation $\bar{u}_N \in G(\bar{\Omega}_N)$, $\bar{u}_N = g_N(\partial\Omega_N)$ proceeds as follows:

(1) *Defect computation*: $\tilde{d}_N = f_N - L_{sp}\tilde{u}_N$.

(2) *Defect correction*: Compute an approximation $\bar{v}_N$ to the exact solution of

$$L_{FD}^N v_N = \tilde{d}_N(\Omega_N), \qquad v_N = 0(\partial\Omega_N). \tag{4.1}$$

(3) *Richardson step*: $\bar{u}_N = \tilde{u}_N + \omega\bar{v}_N$ with a parameter ω.

As the defect correction requires homogeneous boundary conditions the relaxation transfers the boundary values of the start approximation to the new approximation. Consequently, if at the beginning of the iteration the exact boundary values are set they are kept until the end. For the given simple Dirichlet problem the described treatment of boundary conditions seems to be advisable since the problems of boundary relaxation [1] are avoided.

Relaxations for spectral problems are only efficient if fast (Fourier) transforms are avalable. For this purpose the Richardson scheme in contrast to other methods, used for finite difference or finite element problems, is well suited.

The necessity of a defect correction becomes already obvious in the one-dimensional case for the operator $Lu = -u''$. The condition number of the spectral matrix increases like $0(N^4)$ and the corresponding multigrid condition number of $0(N^2)$ is unacceptable. For preconditioning with the three point star, Haldenwang *et al.* [7] have computed the resulting eigenvalues. The extreme ones are $\lambda_{\min} = 1$, $\lambda_{\max} = N(N-1)\sin^2(\pi/2N)$ and hence $\lambda_{\max}$ approaches $\pi^2/4$ for increasing N.

However, the defect correction should not be too expensive. In this context, finite difference and finite element methods are recommended.

4.1. *Smoothing Analysis*

A rigorous local mode analysis as for periodic problems (see Brandt *et al.* [2]) is not possible since the eigenvectors of the spectral matrix are not known. Nevertheless a local consideration of the spectral operator applied to the Chebyshev grid functions yields a useful insight. Let $t_{k,l;N} \in G(\bar{\Omega}_N)$ $(k, l = 0, ..., N)$ be given with components $t^{i,j}_{k,l;N} = t_k(x_i)\, t_l(y_j)$ $(i, j = 0, ..., N)$ $(t_k(x) = \cos(k \arccos x))$. When the coefficients a and b are constant we get in $(i, j) \in I_{\Omega_N}$

$$
\begin{aligned}
(L_{sp} t_{k,l;N})^{i,j} = &\left(\frac{a}{\sin^2(i\pi/N)} k^2 + \frac{b}{\sin^2(j\pi/N)} l^2\right) t_k(x_i)\, t_l(y_j) \\
&- \left(a \cos\frac{i\pi}{N} \Big/ \sin^3\left(\frac{i\pi}{N}\right)\right) k \tilde{t}_k(x_i)\, t_l(y_j) \\
&- \left(b \cos\frac{j\pi}{N} \Big/ \sin^3\left(\frac{j\pi}{N}\right)\right) l t_k(x_i)\, \tilde{t}_l(y_j)
\end{aligned}
$$

with $\tilde{t}_l(x) = \sin(l \arccos x)$ $(l = 0, ..., N)$. It becomes obvious that the coefficients in front of the global dominant terms k^2 and l^2 may locally attain quite different values, in so far as the spectral problem is anisotropic and the weighting of direction changes locally. The same observation can be made for the finite difference operator. Here we get

$$
\begin{aligned}
(L^N_{FD} t_{k,l;N})^{i,j} = &\left(\frac{a}{\sin^2(i\pi/N) - \sin^2(\pi/2N)} \left(\frac{1 - \cos(k\pi/N)}{1 - \cos(\pi/N)}\right)\right. \\
&\left. + \frac{b}{\sin^2(j\pi/N) - \sin^2(\pi/2N)} \left(\frac{1 - \cos(l\pi/N)}{1 - \cos(\pi/N)}\right)\right) t_k(x_i)\, t_l(y_j) \\
&- \frac{a \cot(i\pi/N)}{\sin^2(i\pi/N) - \sin^2(\pi/2N)} \left(\frac{\sin(k\pi/N)}{\sin(\pi/N)}\right) \tilde{t}_k(x_i)\, t_l(y_j) \\
&- \frac{b \cot(j\pi/N)}{\sin^2(j\pi/N) - \sin^2(\pi/2N)} \left(\frac{\sin(l\pi/N)}{\sin(\pi/N)}\right) t_k(x_i)\, \tilde{t}_l(y_j).
\end{aligned}
$$

Therefore an alternating smoothing technique for defect correction seems to be useful. For an exact defect correction the quotient of the above dominant terms determines the spectrum of eigenvalues. By "freezing" it at a certain fixed point of Ω_N the asymptotic bounds 1 and $\pi^2/4$ can be derived. If only high frequencies are taken into account this interval can be reduced somewhat but the parameters then depend on a, b locally and the resulting improvement is small.

4.2. *Richardson Relaxation Techniques*

4.2.1. *Stationary Richardson (SR) and Nonstationary Richardson (NSR) Relaxation*

For the SR relaxation the parameter ω is the same for all sweeps. It is known that the optimal parameter ω_{opt} and smoothing rate μ_{opt} are

$$\omega_{\text{opt}} = \frac{8}{\pi^2+4} \doteq 0.5768, \qquad \mu_{\text{opt}} = \frac{\pi^2-4}{\pi^2+4} \doteq 0.4232.$$

For the NSR relaxation [13] with k different parameters the optimal ones are

$$\omega_{l,\text{opt}} = \left(\frac{\pi^2+4}{8} + \frac{\pi^2-4}{8}\cos\left(\frac{2l-1}{2k}\pi\right)\right)^{-1} \quad (l=1, \ldots, k)$$

with smoothing rate

$$\mu_{k,\text{opt}} = \left| t_k\left(\frac{\pi^2+4}{\pi^2-4}\right)\right|^{-1/k}.$$

Table I shows $\mu_{k,\text{opt}}$ for $k=1, 2, 3, 4, 5$. In the applications we choose $k=3$ since the additional work for greater k is not paying.

4.2.2. *Minimal Residual Richardson (MRR) Relaxation*

Canuto and Quarteroni [3] have already tested the MRR relaxation as an iterative method. We examine its smoothing properties in connection with spectral multigrid methods. The parameter ω is chosen in order to minimize the residual $\bar{d}_N = f_N - L_{sp}\bar{u}_N$ of the new approximation $\bar{u}_N$ in the norm $\|\cdot\|$, defined by

$$\|v_N\| = (v_N, v_N)^{1/2} \qquad \text{with} \quad (v_N, w_N) = N^{-2}\sum_{i,j=1}^{N-1} v_N^{i,j} w_N^{i,j}.$$

TABLE I

Smoothing Rate $\mu_{k,\text{opt}}$ for $k=1, 2, 3, 4, 5$

k	1	2	3	4	5
$\mu_{k,\text{opt}}$	0.4232	0.3136	0.2797	0.2640	0.2550

If a start approximation $\tilde{u}_N$ with the residual $\tilde{d}_N = f_N - L_{sp}\tilde{u}_N$ is given, one iteration step proceeds as follows:

(1) *Defect computation*: Compute the correction $\bar{v}_N$ as in (4.1).

(2) *Parameter computation*: $\omega = (\tilde{d}_N, w_N)/(w_N, w_N)$ with $w_N = L_{sp}\bar{v}_N$.

(3) *Richardson step*: Compute the new approximation $\bar{u}_N$ and residual $\bar{d}_N$:

$$\bar{u}_N = \tilde{u}_N + \omega\bar{v}_N, \qquad \bar{d}_N = \tilde{d}_N - \omega w_N.$$

In comparison with the relaxation in Section 4.2.1 the additional work of computing the adaptive parameters becomes obvious. In the numerical experiments (Sect. 5) this choice of parameters yields good convergence rates also for problems being far removed from Poisson's equation.

4.3. *Defect Correction*

In previous works [12, 19, 20] above all incomplete LU (ILU)-decompositions of $\hat{L}_{FD}$ have been proposed. In [12] the ILU-decomposition with seven diagonals turned out to be the best preconditioner. Furthermore the local mode analysis in Section 4.1 suggests a correction which smoothes in an alternating manner. Therefore we propose the use of alternating ILU-decompositions or alternating zebra relaxation (AZR). AZR was already recommended by Brandt *et al.* [2] for periodic problems with anisotropicity.

4.3.1. *Incomplete LU (ILU)-Decompositions*

Different LU-decompositions result from different numeration techniques in constructing the matrix $\hat{L}_{FD}$ from the difference operator $\hat{L}^N_{FD}$. If we numerate linewise beginning at $(1, 1) \in I_{\Omega_N}$ we call it ILUWN; if we numerate columnwise beginning at $(N-1, 1) \in I_{\Omega_N}$ we call it ILUNO. For the star $L^{i,j}_{FD}$ as in (3.1) the stars $L^{i,j}_D$, $U^{i,j}_D$, and $R^{i,j}_D$ of the decomposition satisfying $\hat{L}^N_{FD} = \hat{L}^N_D\hat{U}^N_D - \hat{R}^N_D$ are deduced. For ILUWN we get

$$L^{i,j}_D = \begin{bmatrix} 0 & 0 & 0 \\ l^{i,j}_{-1,0} & l^{i,j}_{0,0} & 0 \\ 0 & l^{i,j}_{0,-1} & l^{i,j}_{1,-1} \end{bmatrix}, \qquad U^{i,j}_D = \begin{bmatrix} u^{i,j}_{-1,1} & u^{i,j}_{0,1} & 0 \\ 0 & 1 & u^{i,j}_{1,0} \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$R^{i,j}_D = \begin{bmatrix} r^{i,j}_{-2,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r^{i,j}_{2,-1} \end{bmatrix},$$

where

$$l^{i,j}_{0,-1} = \gamma^{i,j}_{0,-1},\ l^{i,j}_{1,-1} = -l^{i,j}_{0,-1} u^{i,j-1}_{1,0},$$
$$l^{i,j}_{-1,0} = \gamma^{i,j}_{-1,0} - l^{i,j}_{0,-1} u^{i,j-1}_{-1,1},$$
$$(l^{i,j}_{0,0})^{-1} = (\gamma^{i,j}_{0,0} - l^{i,j}_{0,-1} u^{i,j-1}_{0,1} - l^{i,j}_{-1,0} u^{i-1,j}_{1,0} - l^{i,j}_{1,-1} u^{i+1,j-1}_{-1,1})^{-1},$$
$$u^{i,j}_{1,0} = (\gamma^{i,j}_{1,0} - l^{i,j}_{1,-1} u^{i+1,j-1}_{0,1})(l^{i,j}_{0,0})^{-1}$$
$$u^{i,j}_{-1,1} = -l^{i,j}_{-1,0} u^{i-1,j}_{0,1} (l^{i,j}_{0,0})^{-1},\ \gamma^{i,j}_{0,1} = \gamma^{i,j}_{0,1} (l^{i,j}_{0,0})^{-1}$$

and

$$r^{i,j}_{2,-1} = l^{i,j}_{1,-1} u^{i+1,j-1}_{1,0},\ r^{i,j}_{-2,1} = l^{i,j}_{-1,0} u^{i-1,j}_{-1,1}.$$

For ILUNO we get

$$L^{i,j}_D = \begin{bmatrix} 0 & 0 & l^{i,j}_{1,1} \\ 0 & l^{i,j}_{0,0} & l^{i,j}_{1,0} \\ 0 & l^{i,j}_{0,-1} & 0 \end{bmatrix}, \qquad U^{i,j}_D = \begin{bmatrix} 0 & u^{i,j}_{0,1} & 0 \\ u^{i,j}_{-1,0} & 1 & 0 \\ u^{i,j}_{-1,-1} & 0 & 0 \end{bmatrix}$$

and

$$R^{i,j}_D = \begin{bmatrix} 0 & 0 & r^{i,j}_{1,2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ r^{i,j}_{-1,-2} & 0 & 0 \end{bmatrix},$$

where

$$l^{i,j}_{1,0} = \gamma^{i,j}_{1,0},\ l^{i,j}_{0,-1} = \gamma^{i,j}_{0,-1} - l^{i,j}_{1,0} u^{i+1,j}_{-1,-1}.$$
$$l^{i,j}_{1,1} = -l^{i,j}_{1,0} u^{i+1,j}_{0,1},$$
$$(l^{i,j}_{0,0})^{-1} = (\gamma^{i,j}_{0,0} - l^{i,j}_{0,-1} u^{i,j-1}_{0,1} - l^{i,j}_{1,0} u^{i+1,j}_{-1,0} - l^{i,j}_{1,1} u^{i+1,j+1}_{-1,-1})^{-1},$$
$$u^{i,j}_{0,1} = \gamma^{i,j}_{0,1} (l^{i,j}_{0,0})^{-1},\ u^{i,j}_{-1,-1} = -l^{i,j}_{0,-1} u^{i,j-1}_{-1,0} (l^{i,j}_{0,0})^{-1},$$
$$u^{i,j}_{0,1} = (\gamma^{i,j}_{0,1} - l^{i,j}_{1,1} u^{i+1,j+1}_{-1,0})(l^{i,j}_{0,0})^{-1}$$

and

$$r^{i,j}_{1,2} = l^{i,j}_{1,1} u^{i+1,j+1}_{0,1},\ r^{i,j}_{-1,-2} = l^{i,j}_{0,-1} u^{i,j-1}_{-1,-1}.$$

In both cases components belonging to boundary points are set to zero. The defect correction with ILUWN or ILUNO proceeds as follows:

(1) Computation of $w_N \in G(\Omega_N)$ by

$$\hat{L}_D^N w_N = \bar{d}_N.$$

(2) Computation of $\bar{v}_N \in G(\bar{\Omega}_N)$, $\bar{v}_N = 0$ $(\partial\Omega_N)$ by

$$U_D^N \bar{v}_N = w_N.$$

If in the difference operator the x-direction (y-direction) prevails also the entries of the rest matrix of ILUWN (ILUNO) become large. Hence if the x-direction (y-direction) becomes dominant the smoothing effect of ILUNO (ILUWN) is utilized. In connection with multigrid methods ILUWN is employed before and ILUNO after coarse grid correction.

Furthermore we remark that "alternating" ILU needs twice the amount of pre-computations. Therefore we recommend to compute the factorizations once and store them.

4.3.2. *Zebra Line Relaxation (ZLR)*

Another appropriate smoother is the alternating direction line relaxation. One first relaxes along lines of constant y and then an analogues sweep along lines of constant x. By solving first for the odd ("white") lines and then for the even ("black") lines we attain the alternating ZLR. After scaling the ith row by $\sin(i\pi/N)$ the tridiagonal linear systems become symmetric and can easily be solved by means of a Cholesky decomposition. A more detailed description of ZLR is given in [1, 2].

5. Convergence

We consider a spectral multigrid algorithm which consists of a relaxation scheme as in Section 4.2 and transfer operators for restriction and interpolation as in [19]. In order to estimate its convergence properties we compute the spectral radius ρ of the multigrid operator by means of the power method [4]. Since MRR yields oscillating rates we present here a suitable mean value. A convergence factor which is related to the work W can be defined by $\rho_W = \rho^{1/W}$. The standard work unit is the amount of work involved in one relaxation sweep on the finest grid. In the numerical computation we use a V-cycle with the grids $\bar{\Omega}_4$, $\bar{\Omega}_8$, $\bar{\Omega}_{16}$, $\bar{\Omega}_{32}$ and fixed numbers ν_1 and ν_2 of relaxations on each grid in the downward and upward branches, respectively. Hence we get $W = (1 + 1/4 + 1/16 + 1/64)(\nu_1 + \nu_2) = 1.328125(\nu_1 + \nu_2)$. ρ_w does not take the total work into account but it should be near the smoothing rate and provide an estimate of efficiency which is independent of both the computer and the programmer.

The following modifications of the multigrid method were investigated:

— type of relaxation: SR, NSR, or MRR

TABLE II

ρ_W for Example (1) Relative to the Cases (α) and (β)

(α)	AZR	ILU2	ILU1	(β)	AZR	ILU2	ILU1
SR	0.9641	0.9090	0.9103	SR	0.4811	0.5120	0.5553
NSR	0.9602	0.8990	0.9005	NSR	0.3993	0.3943	0.4452
MRR	0.8305	0.8766	0.8000	MRR	0.3927	0.4256	0.4001

— type of defect correction: AZR, ILU1 (= ILUWN) or ILU2 (= ILUWN/ILUNO).

Depending on the above modifications we have made good experience with different numbers of sweeps:

$$\nu_1 = \nu_2 = 2 \quad \text{for SR and MRR}$$

$$\nu_1 = 3, \nu_2 = 0 \quad \text{for NSR with AZR}$$

$$\nu_1 = 3, \nu_2 = 3 \quad \text{for NSR with ILU1, ILU2.}$$

We tested examples with coefficients a, b given by

(1) $a(x, y) = b(x, y) = 1$

(2) $a(x, y) = b(x, y) = 1 + \varepsilon \exp(\cos(\pi(x + y)))$

(3) $a(x, y) = 1 + \varepsilon \exp(x)$, $b(x, y) = 1 + \varepsilon \exp(y)$,

where $\varepsilon \in \{0.1, 0.2, 0.3\}$.

TABLE III

ρ_W for Example (2)

Relaxation	Defect correction	$\varepsilon = 0.1$	$\varepsilon = 0.2$	$\varepsilon = 0.3$
SR	AZR	0.5546	0.6552	0.7382
	ILU2	0.5623	0.6571	0.7388
	ILU1	0.6564	0.7543	0.8381
NSR	AZR	0.4724	0.6030	0.7038
	ILU2	0.4142	0.5848	0.6947
	ILU1	0.5966	0.7191	0.8179
MRR	AZR	0.4198	0.4471	0.4769
	ILU2	0.4310	0.4668	0.4790
	ILU1	0.4249	0.5295	0.5331

The problems considered in (1) and (2) are isotropic whereas the problems considered in (3) are anisotropic. Here ε measures the departure of the coefficients from constant and in (3) also the amount of anisotropicity, with $(a/b)_{\max} = (b/a)_{\max} = (1+\varepsilon e)/(1+\varepsilon e^{-1})$. For example (1) the rates ρ_W are listed in Table II relative to the cases

(α) relaxation without coarse grid correction

(β) multigrid iteration (V-cycle).

Of course the multigrid method yields considerably better results than relaxation only. For the V-cycle with SR or NSR relaxation the alternating defect correction AZR and ILU2 are superior to ILU1. The MRR relaxation leads to rates being similar to the NSR relaxation where the roles of ILU1 and ILU2 have changed. The rates ρ_W for the examples (2) and (3) are listed in the Tables III and IV. In example (2) the oscillation of the coefficients leads to quite bad rates for increasing ε. This could only slightly be improved by "filtering" the coefficients [20]. Once more the reported results substantiate the usefulness of alternating defect corrections.

For the MRR relaxation this is not true in general. The corresponding choice of parameters seems to be so good that the smoothing effect cannot be improved. Above all it is remarkable that the rates hardly deteriorate for increasing ε. The MRR parameters are better suited for non-Laplacian operators than the others. This is due to the fact that the parameters of the SR and NSR relaxation are determined by a smoothing analysis of the Laplace operator. In the isotropic case the rates for AZR and ILU2 are nearly identical whereas in the anisotropic case AZR yields the best results. The reason for this observation is due to the fact that AZR needs one and ILU2 two sweeps for smoothing in an alternating manner. Concern-

TABLE IV

ρ_W for Example (3)

Relaxation	Defect correction	$\varepsilon = 0.1$	$\varepsilon = 0.2$	$\varepsilon = 0.3$
SR	AZR	0.5132	0.5386	0.5592
	ILU2	0.5446	0.5706	0.5919
	ILU1	0.5858	0.6101	0.6301
NSR	AZR	0.3936	0.4293	0.4654
	ILU2	0.4256	0.4711	0.5048
	ILU1	0.4955	0.5319	0.5602
MRR	AZR	0.4025	0.4044	0.4130
	ILU2	0.4086	0.4124	0.4184
	ILU1	0.4091	0.4137	0.4188

ing the work of decomposition and correction the AZR- and ILU-techniques are comparable. The alternating ILU needs twice as much work for the decomposition. If the decompositions are stored once this is not very costly.

Altogether we conclude that the MRR relaxation using AZR is the best choice for all examples tested. If L is not too far removed from the Laplace operator the NSR relaxation also yields satisfying results. Furthermore this work shows that alternating defect corrections improve the convergence of spectral methods for Dirichlet problems.

6. Examples

For some examples introduced by Haidvogel and Zang [6] we investigate the convergence properties first of the V-cycle itself and second of the V-cycle within a full multigrid (FMG) iteration [14]. The number of cycles needed to reach the respective accuracy of the discretization is computed numerically. For the FMG iteration we used the same number of V-cycles on all grids. The FMG interpolation is the same as for the V-cycle. Furthermore we employ the NSR relaxation and AZR for defect correction.

By executing a sufficient number of multigrid steps we compute the absolute discretization errors between the exact solution u and the calculated solution u_N using the norms

$$E2 = \|u_N - u\| \qquad \text{and} \qquad EM = \|u_N - u\|_{\max}.$$

Hereby $\|\cdot\|_{\max}$ denotes the maximum norm (on Ω_N). Then we count the number IT of V-cycles needed in order to achieve an accuracy of $\|u_N^{(IT)} - u_N\| < E2$ for the new approximation $u_N^{(IT)}$. IT_V denotes IT for the V-cycle only; IT_{FMG} denotes IT within FMG.

In all cases we started with the zero grid function. Now the convergence rate can be measured by the mean value [2] $\bar{\rho} = (\|u_N^{(IT)} - u_N\| / \|u_N\|)^{1/IT}$. The corresponding convergence factor per work unit is given by $\bar{\rho}_W = \bar{\rho}^{1/W}$. Instead of $\bar{\rho}_W$ we write $\bar{\rho}_W^V$ for the V-cycle and $\bar{\rho}_W^{\mathrm{FMG}}$ for the FMG iteration.

We examine certain Poisson equations introduced in [6] which were solved by means of Lanczos' tau method [10].

Example (1):

$$-\Delta u(x, y) = 32\,{}^2\sin(4\pi x)\sin(4\pi y) \quad ((x, y) \in \Omega)$$
$$u(x, y) = 0 \quad ((x, y) \in \partial\Omega).$$

Example (2):

$$-\Delta u(x, y) = 1 \quad ((x, y) \in \Omega)$$
$$u(x, y) = 0 \quad ((x, y) \in \partial\Omega).$$

Example (3):

$$-\Delta u(x, y) = w(x)\, v(y) + v(x)\, w(y) \quad ((x, y) \in \Omega)$$
$$u(x, y) = 0 \quad ((x, y) \in \partial\Omega),$$

where

$$v(x) = \begin{cases} -\frac{1}{4}(x+1), & x < 0 \\ \frac{1}{2}x^2 - \frac{1}{4}x - \frac{1}{4}, & x \geqslant 0 \end{cases}$$

and

$$w(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geqslant 0. \end{cases}$$

The results for the examples are presented in Table V.

The three examples differ in the smoothness of the exact solution. The solution of the first example is analytic whereas the others have singularities. In example (2) the irregularity occurs near the four corners and in example (3) along both coordinate axes. A more detailed discussion of the examples is given in [6]. The num-

TABLE V

Numerical Results for the Examples in Section 6

Example (1)

N	$E2$	EM	IT_V	$\bar{\rho}_W^V$	IT_{FMG}	$\bar{\rho}_W^{\mathrm{FMG}}$
8	1.66	6.36	1	0.3208	1	0.3877
16	2.23E−3	5.25E−3	2	0.3648	2	0.4564
32	7.74E−13	2.17E−12	7	0.3742	7	0.4743

Example (2)

N	$E2$	EM	IT_V	$\bar{\rho}_W^V$	IT_{FMG}	$\bar{\rho}_W^{\mathrm{FMG}}$
8	7.00E−6	1.61E−5	3	0.3543	2	0.2360
16	1.73E−7	7.47E−7	4	0.3432	2	0.2011
32	5.31E−9	5.51E−8	5	0.3595	2	0.1547

Example (3)

N	$E2$	EM	IT_V	$\bar{\rho}_W^V$	IT_{FMG}	$\bar{\rho}_W^{\mathrm{FMG}}$
8	2.83E−4	1.00E−3	2	0.3234	2	0.3589
16	6.52E−5	2.82E−4	2	0.3403	2	0.3983
32	1.69E−5	9.05E−5	2	0.3336	2	0.3998

bers IT_V, IT_{FMG} depend on the respective accuracy. For example (2) the FMG iteration is very effective since the lower level grids give good approximations to the finer grids.

A direct comparison with the tau method in [6] shows that the pseudospectral discretization yields a higher accuracy.

7. Chebyshev–Fourier Problems

We consider two kinds of elliptic problems. The first one is the *rectangular Chebyshev–Fourier problem.* For given functions a, b, f it is given by

$$\frac{\partial}{\partial x}\left(a\frac{\partial}{\partial x}u\right)+\frac{\partial}{\partial y}\left(b\frac{\partial}{\partial y}u\right)=f \qquad \text{on} \quad \Omega=(-1,1)\times(0,2\pi) \tag{7.1}$$

with combined Dirichlet and periodic boundary conditions.

The second one is the *annular Chebyshev–Fourier problem*, given by

$$\frac{\partial}{\partial r}\left(ra\frac{\partial}{\partial r}u\right)+\frac{\partial}{\partial \vartheta}\left(\frac{1}{r}a\frac{\partial}{\partial \vartheta}u\right)f \qquad \text{on} \quad G=(r_0,r_1)\times(0,2\pi), \tag{7.2}$$

where $1\leqslant r_0<r_1<\infty$, with radial boundary conditions, i.e., Dirichlet at $r=r_1$ and Neumann (or Direchlet) at $r=r_0$. In ϑ we prescribe periodic boundary conditions. Such problems arise from annular Poisson equations which are transformed using polar coordinates (r,ϑ). By means of a change of variable we get $\partial/\partial r=(2/(r_1-r_0))(\partial/\partial x)$ for $r\in(r_0,r_1)$, $x\in(-1,1)$ and it is sufficient to consider problem (7.1).

For the discretization we use in x resp. r Chebyshev polynomials and in y resp. ϑ trigonometric polynomials. The corresponding grid points are given by

$$(x_i,y_j)=\left(\cos\left(\frac{i\pi}{N}\right),\frac{2j\pi}{N}\right)$$

resp.

$$(r_i,\vartheta_j)=\left(\frac{r_0+r_1}{2}+\frac{r_1-r_0}{2}\cos\left(\frac{i\pi}{N}\right),\frac{2j\pi}{N}\right) \qquad \text{for} \quad i,j=0,\ldots,N.$$

The way of discretization is straightforward and the details can be taken from Zang *et al.* [19]. In contrast to [19] we use in the direction of periodicity the "midpoint discretization" introduced by Brandt *et al.* [2]. By this approach the information in the highest mode is retained and an improved accuracy can be attained. Moreover we treat the Dirichlet (Neumann) boundary conditions in an explicit manner since the implicit treatment gives much larger spectra of eigenvalues (see [3]).

TABLE VI

ρ_W for Problem (7.1)

ε	0.0	0.1	0.2
SR	0.5049	0.5734	0.6354
NSR	0.4105	0.5039	0.5672
MRR	0.4326	0.4539	0.4844

The smoothing analysis shows that for isotropic problems the direction of Direchlet boundary conditions is dominant and an exclusive smoothing into this direction is enough. A further step in azimuth gives only an improvement for anisotropic problems or when there are many more grid points in the Fourier direction than in the Chebyshev one (such as for the airfoil problem studied in [13]).

Since ZLR yields similarly good results as ILU but is not so costly in calculating the decompositions we are concerned with it furthermore. For problem (7.1) it consists of relaxing along lines of constant x by solving

$$\frac{a(1/2(x_i + x_{i-1}), y_j)}{s_{i-1/2}s_i}\bar{v}^N_{i-1,j} - \left(\frac{a(1/2(x_i + x_{i-1}), y_j)}{s_{i-1/2}s_i} + \frac{a(1/2(x_i + x_{i+1}), y_j)}{s_{i+1/2}s_i} + \frac{b(x_i, y_{j-1/2})}{(2\pi/N)^2} + \frac{b(x_i, y_{j+1/2})}{(2\pi/N)^2}\right)\bar{v}^N_{i,j} + \frac{a(1/2(x_i + x_{i+1}), y_j)}{s_{i+1/2}s_i}\bar{v}^N_{i+1,j}$$
$$= r^N_{i,j} - \frac{b(x_i, y_{j-1/2})}{(2\pi/N)^2}v^N_{i,j-1} - \frac{b(x_i, y_{j+1/2})}{(2\pi/N)^2}v^N_{i,j+1}$$

with $s_{i\pm 1/2} = \sin((i \pm 1/2)\pi/N)$, $s_i = \sin(i\pi/N)$ for the gird function $\bar{v}^N$. v^N denotes the old grid approximation. Vectorization is achieved by solving first for the odd ("white") and then for the even ("black") lines, resulting in ZLR. After scaling the ith row by s_i the tridiagonal systems become symmetric and can easily be solved by

TABLE VII

ρ_W for Problem (7.2) with Mixed Dirichlet and Periodic Boundary Conditions

ε	0.0	0.1	0.2
SR	0.5033	0.5705	0.7043
NSR	0.4172	0.4781	0.6472
MRR	0.4341	0.4581	0.4820

TABLE VIII

ρ_W for Problem (7.2) with Mixed Dirichlet/Neumann and Periodic Boundary Conditions

ε	0.0	0.1	0.2
SR	0.5142	0.6040	0.6715
NSR	0.4504	0.5231	0.6158
MRR	1.0000	1.0000	1.0000

means of a Cholesky decomposition. In the case of Neumann boundary conditions at $r = r_0$ the left sided difference quotients is given by

$$(1/2 \sin^2(\pi/2N))(\bar{v}^N_{N-1,j} - \bar{v}^N_{N,j}) = r^N_{N,j}.$$

After a suitable scaling of this row the corresponding tridiagonal systems also become symmetric. For problem (7.1) resp. (7.2) we tested examples with variable coefficients, given by

$$a(x, y) = b(x, y) = 1 + \varepsilon \exp(\cos(\beta(\pi x + y)))$$

resp.

$$a(r, \vartheta) = 1 + \varepsilon \exp(\cos(\beta(r + \vartheta)))$$

for $\beta = 10\varepsilon$, $\varepsilon \in \{0.0, 0.1, 0.2\}$ and $r_0 = 1$, $r_1 = 5$. The rates ρ_W for problem (7.1) are listed in Table VI. Table VII resp. Table VIII contain the results for problem (7.2) with mixed Dirichlet resp. Dirichlet/Neumann and periodic boundary conditions.

The results show the improvements by NSR and MRR relaxation. MRR relaxation yields also good results for problems far removed from Poisson's

TABLE IX

Numerical Results for the Examples (1)–(3) in Section 7

Example	N	EM	$E2$	IT_V	ρ_W
1	8	$8.14E-2$	$3.39E-2$	1	0.3215
	16	$1.21E-4$	$4.19E-5$	3	0.3740
	32	$2.36E-12$	$8.92E-13$	8	0.3954
2	8	$6.96E-2$	$2.37E-2$	1	0.3464
	16	$9.49E-5$	$3.67E-5$	3	0.3538
	32	$2.25E-12$	$7.33E-13$	8	0.3847
3	8	$2.39E-1$	$6.87E-2$	2	0.4532
	16	$1.09E-4$	$3.09E-5$	4	0.4731
	32	$2.36E-12$	$7.99E-13$	10	0.4751

equation. The convergence behaviour is stable for increasing ε and the rates are less than 0.5 for the problems in Tables VI and VII. In the case of Neumann boundary conditions the rate is equal to one due to the fact that the symmetric part of the spectral matrix is indefinite [3].

For some examples introduced by Streett *et al.* [13] we show in Table IX the accuracy of spectral methods. We consider the Poisson equation, i.e., $a = b = 1$, and

(1) $u(x, y) = \sin(\pi x + \pi/4) \sin(\pi \cos(y) + \pi/4)$ for problem (7.1).

(2) $u(r, \vartheta) = \sin(\pi r) \sin(\pi \cos(\vartheta) + \pi/4)$ for problem (7.2) with Dirichlet boundary conditions at $r = 1$ and $r = 5$.

(3) $u(r, \vartheta) = \cos(\pi r) \sin(\pi \cos(\vartheta) + \pi/4)$ for problem (7.2) with Neumann boundary conditions at $r = 1$.

The numerical results are given for the V-cycle with NSR relaxation. The notations are in common with Table V.

We remark that a direct comparison with the convergence rates in [13] is not possible. Streett *et al.* [13] used the "equivalent smoothing rate" which takes into account all the work of the multigrid method.

Altogether the results presented further substantiate the usefulness of line relaxation in spectral multigrid methods. It is obvious that these techniques can also be successfully adopted to the more complicated problems of transonic flow.

ACKNOWLEDGMENT

I acknowledge helpful discussions with Professor Kristian Witsch, Düsseldorf.

REFERENCES

1. A. BRANDT, in *Proceedings of the Conference Held at Köln-Porz, West Germany*, 1981, edited by A. Dold and B. Eckmann (Springer-Verlag, New York, 1982).
2. A. BRANDT, S. R. FULTON, AND G. D. TAYLOR, *J. Comput. Phys.* **58**, 96 (1985).
3. C. CANUTO AND A. QUARTERONI, *J. Comput. Phys.* **60**, 315 (1985).
4. D. K. FADDEJEW AND W. M. FADDEJEWA, *Numerische Methoden der linearen Algebra* (R. Oldenbourg Verlag, München–Wien, 1979).
5. D. GOTTLIEB AND S. A. ORSZAG, "NSF-CBMS Monograph No. 26," Soc. Ind. and Appl. Math., Philadelphia, 1977.
6. D. B. HAIDVOGEL AND T. ZANG, *J. Comput. Phys.* **30**, 167 (1979).
7. P. HALDENWANG, G. LABROSSE, S. ABBOUDI, AND M. DEVILLE, *J. Comput. Phys.* **55**, 115 (1984).
8. W. HEINRICHS, Thesis, University of Düsseldorf, 1986 (unpublished).
9. P. W. HEMKER, *Boundary and Interior Layers, Computational and Asymptotic Methods*, J. J. H. Miller, ed. (Boole Press, Dublin, 1980).
10. C. LANCZOS, *Applied Analysis* (Prentice–Hall, Englewood Cliffs, NJ, 1956).
11. S. A. ORSZAG, *J. Comput. Phys.* **37**, 70 (1980).
12. T. N. PHILLIPS, T. A. ZANG, AND M. Y. HUSSAINI, *Multigrid Methods for Integral and Differential Equations*, D. J. Paddon and H. Holstein, eds. (Clarendon Press, Oxford, 1985).

13. C. L. STREETT, T. A. ZANG, AND M. Y. HUSSAINI, *J. Comput. Phys.* **57**, 43 (1985).
14. K. STÜBEN AND U. TROTTENBERG, in *Proceedings of the Conference Held at Köln-Porz, West Germany*, 1981, edited by A. Dold and B. Eckman (Springer-Verlag, New York, 1982).
15. C. TEMPERTON, *J. Comput. Phys.* **31**, 1 (1979).
16. C. TEMPERTON, *J. Comput. Phys.* **34**, 314 (1980).
17. C. A. THOLE, Thesis, University of Bonn, 1983 (unpublished).
18. R. S. VARGA, *Matrix Iterative Analysis* (Prentice–Hall, Englewood Cliffs, NJ, 1962).
19. T. A. ZANG, Y. S. WONG, AND M. Y. HUSSAINI, *J. Comput. Phys.* **48**, 485 (1982).
20. T. A. ZANG, Y. S. WONG, AND M. Y. HUSSAINI, *J. Comput. Phys.* **54**, 489 (1984).